Matrix method to solve the differential power flow equation in the frequency domain

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Abstract: We present a method to obtain the frequency response of step index (SI) plastic optical fibers (POFs) based on the power flow equation generalized to incorporate the temporal dimension where the fiber diffusion and attenuation are functions of the propagation angle. To solve this equation we propose a fast implementation of the finite-difference method in matrix form. The model provides the space-time evolution of the angular power distribution when it is transmitted throughout the fiber which gives a detailed picture of the POFs capabilities for information transmission.

Introduction. In a previous work, we devised a method based on Gloge's equation and on experimental far field patterns (FFPs) to obtain the angular diffusion and attenuation functions characteristic of a given fiber[1] that account for the fiber spatial behavior. However, as the temporal dependence is not explicit in the equation, the frequency response and bandwidth cannot be calculated from this approach. Thus, in this paper we present a fast and robust matrix approach of the finite-difference method to solve the temporal generalization of the power flow equation in the frequency domain for the given angular diffusion and attenuation functions[2,3]. The frequency responses for a given fiber at a range of lengths calculated with our method were used to validate it by comparing them to our own experimental measurements for the same fibers[4]. Thus, we showed that the shape of the measured frequency responses is far from Gaussian and is, however, well reproduced by our model predictions. Moreover, the proposed model provides the space-time evolution of the optical power when it is transmitted throughout the fiber. This space-time function gives a complete description of power evolution and can be used to derive parameters with strong impact over fiber transmission capabilities and to extend model predictions where it is difficult or unpractical to measure them, giving a full insight of fiber behavior. Thus, in the paper we first describe our method and then, we discuss how the information provided by the space-time power distribution can be used to understand and improve fiber transmission behavior.

Matricial approach proposed to solve the space-time power flow equation. We use Gloge's power flow equation to describe the evolution of the modal power distribution as it is transmitted throughout a POF where different modes are characterized by their inner propagation angle with respect to fiber axis (θ), which can be taken as a continuous variable[1]. We make no assumptions about the angular diffusion and attenuation which are described as functions of θ, $d(\theta)$ and $\alpha(\theta)$ respectively. Following the procedure described by Gloge[2] to introduce the temporal dimension, the partial derivative of the optical power, $P(\theta,z,t)$, with respect to z, we get the following equation:

$$\frac{\partial P(\theta,z,t)}{\partial z} = -\alpha(\theta)P(\theta,z,t) - \frac{n}{c\cos\theta}\cdot\frac{\partial P(\theta,z,t)}{\partial t} + \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\cdot d(\theta)\cdot\frac{\partial P(\theta,z,t)}{\partial\theta}\right). \tag{1}$$

Then, we take the Fourier transform at both sides of Eq. (1) and use the Fourier derivation property to obtain the following simplified equation:

$$\frac{\partial p(\theta,z,\omega)}{\partial z} = -\left(\alpha(\theta) + \frac{n}{c\cos\theta}\cdot j\omega\right)p(\theta,z,\omega) + \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\cdot d(\theta)\cdot\frac{\partial p(\theta,z,\omega)}{\partial\theta}\right), \tag{2}$$

where $p(\theta,z,\omega)$ is the Fourier transform of $P(\theta,z,t)$. To solve this differential equation we implement a finite-difference method where we use a forward difference for the first z derivative, and a first and second-order central differences for the first and second angular derivatives respectively. Thus, the power at angle θ and distance $z+\Delta z$ is obtained as the linear combination of the power at the same angle and the two adjacent angles $(\theta+\Delta\theta, \theta-\Delta\theta)$ for a distance z as shows the following equation:

$$
\begin{aligned}
p(\theta, z+\Delta z, \omega) \quad &= \left(1-\left(\alpha(\theta)+\frac{n}{c\cos\theta}\cdot j\omega\right)\Delta z\right)p(\theta, z, \omega) \\
&+\frac{\Delta z}{2\cdot\Delta\theta}\left(\frac{d(\theta)}{\theta}+d'(\theta)\right)\left(p(\theta+\Delta\theta, z, \omega)-p(\theta-\Delta\theta, z, \omega)\right) \\
&-\frac{2d(\theta)\Delta z}{\Delta\theta^2}\,p(\theta, z, \omega) \\
&+\frac{d(\theta)\Delta z}{\Delta\theta^2}\left(p(\theta+\Delta\theta, z, \omega)+p(\theta-\Delta\theta, z, \omega)\right).
\end{aligned}
\tag{3}
$$

Equation (3) can be expressed in a more compact representation in matrix form. In fact, the differential changes in the angular power distribution at each Δz step are given by a simple matrix product. Thus, given the angular power at an initial length z_1, the power distribution at a longer length z_2 can be calculated with the following matrix equation:

$$
\mathbf{p}(z_2, \omega) = \left(\mathbf{A}(\omega)+\mathbf{D}\right)^m \cdot \mathbf{p}(z_1, \omega), \tag{4}
$$

where $\mathbf{p}$ is a vector where each component k is the power at the discretized propagation angle $\theta = k\cdot\Delta\theta$, and $m = \frac{z_2 - z_1}{\Delta z}$ is an integer that can be found for any pair of lengths, $z_2 > z_1$ providing we choose a small Δz. **A** is a diagonal matrix that describes power propagation without diffusion. Its elements are obtained from Eq. (3) as

$$
A_{k,k}(\omega) \approx 1-\Delta z\cdot\alpha(k\cdot\Delta\theta)-\Delta z\cdot\frac{n}{c\cos(k\cdot\Delta\theta)}\cdot j\omega \tag{5}.
$$

Notice that **A** is the only frequency dependent term in Eq. (4). Iteration over the values of ω gives the complete spatial and temporal evolution of the optical power in the fiber. The complex values of $A_{k,k}(\omega)$ are obtained by sampling the angular frequency ω as required for a precise calculation of the inverse discrete Fourier transform of $p(\theta, z, \omega)$ to obtain $P(\theta, z, t)$. The matrix **D** is a tri-diagonal matrix which accounts for diffusion along the fiber. Its elements for $k > 0$ are:

$$
\begin{aligned}
D_{k,k-1} \quad &= \left(d(k\cdot\Delta\theta)-\frac{1}{2}\frac{d(k\cdot\Delta\theta)}{k}-\frac{1}{2}d'(k\cdot\Delta\theta)\Delta\theta\right)\frac{\Delta z}{\Delta\theta^2} \\
D_{k,k} \quad &= -2d(k\cdot\Delta\theta)\frac{\Delta z}{\Delta\theta^2} \\
D_{k,k+1} \quad &= \left(d(k\cdot\Delta\theta)+\frac{1}{2}\frac{d(k\cdot\Delta\theta)}{k}+\frac{1}{2}d'(k\cdot\Delta\theta)\Delta\theta\right)\frac{\Delta z}{\Delta\theta^2}.
\end{aligned}
\tag{6}
$$

These matrix elements describe power diffusion through a differential fiber length giving the fraction of the power that flows out from a given angle and that drifting to this angle from the adjacent ones.

To obtain the boundary condition at $k=0$, corresponding to $\theta = 0$, we use the approximation proposed in[5], and the fact that $P(\theta)$ is an even function of the angle, obtaining

$$
D_{0,0} = -4d(0)\frac{\Delta z}{\Delta\theta^2} \qquad D_{0,1} = 4d(0)\frac{\Delta z}{\Delta\theta^2}. \tag{7}
$$

The other boundary condition at the maximum $k = N$ is given by Eq. (6). Thus, so that there are no losses due to diffusion, the value of $D_{N,N-1}$ must compensate for the absence of the term $D_{N,N+1}$ such that the sum of terms is zero, resulting in

$$
D_{N,N-1} = 2d(N)\frac{\Delta z}{\Delta\theta^2} \qquad D_{N,N} = -2d(N)\frac{\Delta z}{\Delta\theta^2}. \tag{8}
$$

Matrix $(\mathbf{A}(\omega)+\mathbf{D})$ carries all space-time information concerning power propagation through the fiber and thus, gives a complete description of the fiber as a transmission system. The key of the method we propose to solve Eq. (2) is to take advantage of the sparse nature of this matrix. Therefore, calculating multiple matrix powers is more efficient than performing the same number of iterations, particularly when using MatLab®. Even more, it is not necessary to re-calculate these matrices when changing the initial condition to obtain the space-time output power distributions, as they only depend on the fiber diffusion and attenuation. The values of Δz and $\Delta\theta$ that are critical for convergence have been determined according to the required precision. In the calculations presented here we have used $\Delta z = 0.001$ m and $\Delta\theta = 0.005$ rad obtaining accurate results. The execution time for these values and a typical simulation of a 150 m fibre length to obtain the power distribution at 5 m steps is 0.03 s, more than 70 times faster than the method used in[1], based on the MatLab® partial derivative equation solver.

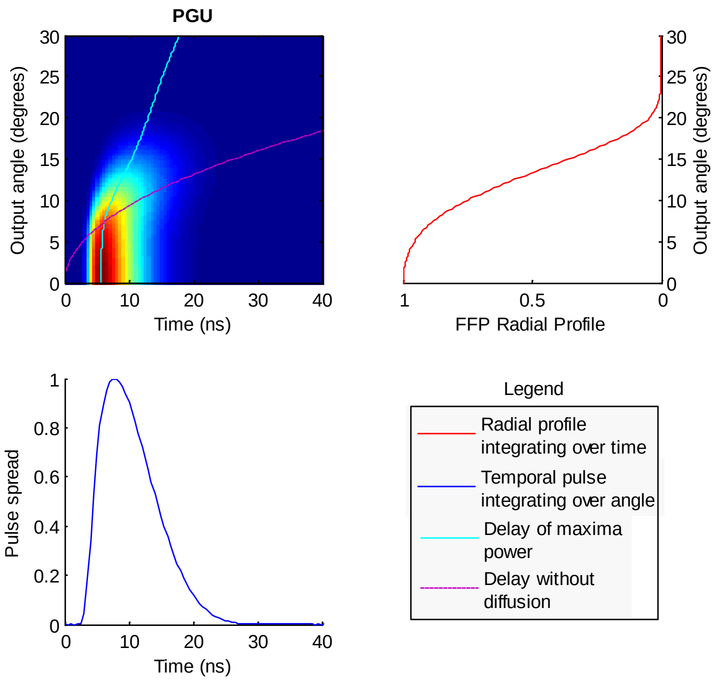

Fig. 1: The graph on the upper left is the image representation of the space-time power distribution at the output of 150 m of the PGU fiber. Below, the lower leftmost graph shows the overall pulse spread obtained as the integral of power over output angle. The power integral over time renders the radial profile of the FFP shown on the upper rightmost graph.

Space-time power distribution. Once the initial condition in vector form is multiplied by the system matrix, it is possible to obtain the power distribution as a function of output angle and frequency $p(\theta,z,\omega)$ at a given length $z=L$ for a given input source. Its inverse Fourier transform gives the power temporal spread for each output angle at this length, $P(\theta,L,t)$ that we call space-time power distribution. In Figure 1, the upper leftmost graph shows the power at the output of a 150 m PGU fiber as an example. Time in nanoseconds is shown on the horizontal axis and output angle in degrees on the vertical axis. Each row represents the temporal pulse arriving at a given output angle. The integrated power over the angle results in the temporal pulse spread, shown normalized in the graph below the image. The global frequency response $H(L,\omega)$ can be obtained as its Fourier transform. The image columns are the radial profiles of the spatial power distribution at fixed times. The integrated power over time gives the radial profile of the FFP represented on the upper rightmost graph as normalized power on the horizontal axis versus angle in degrees on the vertical axis. These simulated radial profiles were used to estimate the fiber attenuation and diffusion along experimental ones[1]. On the image, the superimposed solid line joins the angular positions at which the maximum power reaches the fiber end at each temporal delay. The dashed line shows the delay as a function of angle, obtained without diffusion which is given by the ray-theory inverse cosine law.

An important aspect revealed by the triple dependence of power with propagation angle, length and time/frequency is that to obtain the pulse spread or the frequency response at one given length L, it is not enough to know the pulse spread or the frequency response at any shorter length. To compute the total acquired delay at a given angle and fibre length it is necessary to know the previous path followed by the power reaching that angle, which implies to know the space-time power distribution: $p(\theta, L_0 < L, t)$. Thus, to be able to calculate the frequency response at any length, it is necessary to know the angular power distribution right at the fibre input: $P(\theta, z = 0, t = 0)$ where there is no propagation acquired temporal delay. In fact, previous experimental results suggest that the input distribution has a strong impact on bandwidth changing the balance of diffusion and differential attenuation[6]. Thus, on the basis of the discrepancies found for short and middle length fibers we found that the initial angular power distribution is critical to predict the temporal behavior. The strong initial diffusion suffered by the optical power when it enters the fibre is introduced into our present framework as an independent effect by means of another matrix, which we called injection matrix $\mathbf{J}$, is very different for each fibre type and was estimated as an arrangement of our experimental radial profiles for 1.25 meter fibres obtained launching a He-Ne laser beam at different angles as described before[6]. Therefore, the matrix product of the injection matrix and the transmitter angular power distribution gives the vector describing the angular power distribution just after entering the fibre[4].

The image in Figure 1 shows that optical power that exits the fiber over a cone from 0º to 8º is concentrated over a relatively narrow time slot. Above this angular range, pulses have a wider time spread and their peaks increase with the output angle as shows the blue line in the image. At these angles, the power peaks are reached at lower times than for the cosine prediction indicating noticeable shorter delays than those that would be obtained in the absence of diffusion. In other words, diffusion improves the fiber transmission capability. The horizontal time shift at the lowest angles does not affect the fiber behavior as it is an overall delay. Therefore, the power exiting the fiber at the highest angles is also that with the longest delays. Thus, an easy way to improve fiber capacity is the use of a spatial filter that removes or attenuates the tail at the higher angles as was experimentally[7] and theoretically[8] shown before. As most power is confined in a range of lower angles, filtering out the power at the highest angles will imply small power loss while producing a narrower overall impulse response.

Conclusions. In summary, we have described a fast and robust method that provides the space-time optical power distribution with length which gives a complete description of the fibre behaviour. Model predictions have been found to reproduce experimental measurements of different POF parameters and can be also applied to improve some propagation properties[7,8]. In addition, the method offers a flexible tool to study the effects of using different devices, such as scramblers, tappers, etc, or the impairments occasioned by defects and imperfections as they can be modelled as matrices to be introduced in our framework[9].

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